ON SOME DYNAMIC PHENOMENA IN FLEXIBLE FIBERS

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The investigation is concerned with dynamic phenomena at the free end of a flexible fiber. A mathematical and energy basis is given for the sudden increase in velocity of points in the neighborhood of the fiber tip.

Consider a flexible, inextensible fiber (a little chain) with a uniform mass distribution along its length l and having its upper end fixed and its lower end free.

Confining ourselves to small in-plane deflections of the fiber from its vertical position, we obtain the equation

$$\frac{\partial^2 v}{\partial t^2} = g \frac{\partial}{\partial s}\left[(l-s)\frac{\partial v}{\partial s}\right] \tag{1}$$

Here v is the horizontal deflection of points; l is the fiber length; s is the running coordinate along the fiber; and t is time.

The initial and boundary conditions for (1) take the form

$$v(s, 0) = f_1(s), \quad \left[\frac{\partial v}{\partial t}\right]_{t=0} = f_2(s), \quad v(0, t) = 0, \quad \int_0^l \left[\frac{\partial}{\partial t}\left(s\frac{\partial v}{\partial t}\right) - gv\right] ds = 0 \tag{2}$$

The last condition is an expression of the moment of momentum theorem for the fiber with respect to its point of suspension, for small horizontal deflections.

The solution of (1) may be written in the form

$$v = \sum_{k=1}^{\infty} v_k(s, t), \quad v_k = [C_{1k}J_0(\sigma) + C_{2k}N_0(\sigma)](A_k \cos\omega_k t + B_k \sin\omega_k t), \quad \sigma = 2\omega_k \sqrt{l - s/g} \tag{3}$$

Here J_0 and N_0 are Bessel functions of the first and second kind; the constants A_k and B_k are determined by initial conditions while the constants C_{1k}, C_{2k} and ω_k are determined from the boundary conditions.

However, it is easily seen from (3) that $v \to \infty$ for $s \to l$ unless all C_{2k} are equal to zero, since $N_0(\sigma) \to \infty$ for $\sigma \to 0$, and therefore (3) violates the original assumption of small v.

In [1], the additional requirement was imposed that v and all its derivatives be finite in the region $0 \leqslant s \leqslant l$, thus leading to the Eq. $C_{2k} = 0$, but the last condition in (2) was ignored. In other words, the solution was obtained within a narrower class of functions, which was completely correct mathematically, but did not completely satisfy the physical conditions of the problem.

Indeed, by means of such a solution it is impossible to explain, for example, the emergence of supersonic velocities at the fiber tip (the whip crack) which may occur even for small disturbances at the point of suspension. At the same time, the utilization of the last Eq. in (2) instead of boundedness requirements leads to an important qualitative conclusion

$$|v(l)| \geqslant \delta > 0 \quad \text{for} \quad f_1(s) \to 0, \qquad f_2(s) \to 0 \quad (0 \leqslant s \leqslant l) \tag{4}$$

where δ is a fixed number such that if (4) would imply $v(l) \to 0$, then this would occur in (3) under the conditions of (2).

Hence, the linearized problem for a suspended fiber leads to infinite solutions at the tip, because the corresponding nonlinear problem is incorrect in the region $0 \leqslant s \leqslant l$; in other words, arbitrarily small disturbances at the point of suspension may lead to sufficiently large disturbances (in particular, velocities) at the free end.

Consider the above phenomenon from an energy point of view. Suppose that an isolated transverse wave of small amplitude were generated at the point of suspension (for example, as a result of a discontinuous horizontal displacement of the suspension point followed by its return to the original position after a time t_0). We will assume that prior to the occurrence of this wave the fiber was at rest in a vertical position.

The total mechanical energy in the fiber is given in this case by Expression

$$E = \frac{1}{2}\rho \int_{s_1}^{s_2} \left[\left(\frac{\partial v}{\partial t}\right)^2 + g(l-s)\left(\frac{\partial v}{\partial s}\right)^2\right] ds = E_0 = \text{const} \tag{5}$$

where s_1 and s_2 are the coordinates of the leading and trailing wave fronts.

Noting that in the undisturbed state

$$T = g\rho(l-s) \tag{6}$$

where T is the tension in the fiber and considering that v and its derivatives are small, we obtain for the propagation velocities of the leading and trailing wave fronts the Formulas

$$\frac{ds_1}{dt} = \lambda_1 = \sqrt{g(l-s_1)}, \qquad \frac{ds_2}{dt} = \lambda_2 = \sqrt{g(l-s_2)} \tag{7}$$

Clearly,

$$s_1 = t\sqrt{gl} - {}^1\!/_4 gt^2, \qquad s_2 = (t-t_0)\sqrt{gl} - {}^1\!/_4 g(t-t_0)^2 \tag{8}$$

and consequently

$$\Delta s = s_1 - s_2 = t_0\left(\sqrt{gl} - {}^1\!/_2 gt\right) \tag{9}$$

It is not difficult to calculate the time interval t_* for the wave to reach the free end

$$t_* = 2\sqrt{l/g} \tag{10}$$

From (9) and (10) it follows that $\Delta s \to 0$ for $t \to t_*$. Then, sufficiently close to the fiber tip, (5) may be approximated by

$$E_0 \approx \frac{1}{2}\rho\left(\left[\frac{\partial v}{\partial t}\right]^2 + g(l-s)\left[\frac{\partial v}{\partial s}\right]^2\right)\Delta s \tag{11}$$

Here $[\partial v/\partial t]^2$ and $[\partial v/\partial s]^2$ are the squares of the jumps averaged over Δs.

Noting that at the front of a strongly discontinuous wave of inextensible fiber the relation

$$\left[\frac{\partial v}{\partial t}\right] = -\lambda\left[\frac{\partial v}{\partial s}\right]$$

holds, where λ is the velocity of propagation of the wave front, then (11) yields $E_0 \approx \rho[\partial v/\partial t]^2\Delta s$ and, consequently,

$$\left[\frac{\partial v}{\partial t}\right] \approx \sqrt{E_0/\rho\Delta s} \to \infty \quad \text{for} \quad t \to t_* \tag{12}$$

where (12) holds for sufficiently small E_0.

Thus, as the isolated wave approaches the free end of the fiber, the length of the wave approaches zero (since the velocity of the trailing wave front is always greater than that of the leading wave front) and the energy E_0 is concentrated in an infinitesimal segment of the fiber adjoining the free end, so that the velocity of that end tends to infinity.

Naturally, this result cannot yield any quantitative information with regard to the phenomenon under investigation, since the solution (3) and conditions (2) are based on the assumption that v and $\partial v/\partial t$ are small. However, in this case the similarity between the physical phenomena in the linear and nonlinear cases is very clear. Indeed, for large v and $\partial v/\partial t$

the tension in the fiber above the wave will be greater than below it (even though this tension may no longer be subject to relation (6)) and, consequently, the velocity of the trailing front will always be greater than that of the leading front. Moreover, even in the nonlinear case $T(l, t) = 0$, i.e. the velocity of the leading front will approach zero. Thus, the concentration of all mechanical energy in a small segment near the free end will occur in the nonlinear case as well, which leads to the sharp increases in the fiber tip velocities actually observed for sufficiently small disturbances.

The foregoing considerations show that artificial conditions guaranteeing the correctness in a problem do not always reflect the true physical occurrence; in the case at hand, the incorrect solutions reflect the physical phenomena in a fiber better than the correct ones.

BIBLIOGRAPHY

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